EPICOMPLETE ARCHIMEDEAN ℓ-GROUPS AND VECTOR LATTICES

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ABSTRACT. An object G in a category is *epicomplete* provided that the only morphisms out of G which are simultaneously epi and mono are the isomorphisms. We characterize the epicomplete objects in the category **Arch**, whose objects are the archimedean lattice-ordered groups (archimedean ℓ-groups) and whose morphisms are the maps preserving both group and lattice structure (ℓ-homomorphisms). Recall that a space is *basically disconnected* if the closure of each cozero subset is open.

Theorem. *The following are equivalent for* $G \in$ **Arch**.
  (a) G *is* **Arch** *epicomplete.*
  (b) G *is an* **Arch** *extremal suboject of* $D(Y)$ *for some basically disconnected compact Hausdorff space* Y. *Here* $D(Y)$ *denotes the continuous extended real-valued functions on* Y *which are finite on a dense subset.*
  (c) G *is conditionally and laterally* σ-*complete (meaning each countable subset of positive elements of* G *which is either bounded or pairwise disjoint has a supremum), and* G *is divisible.*

The analysis of **Arch** rests on an analysis of the closely related category **W**, whose objects are of the form (G, u), where $G \in$ **Arch** and u is a *weak unit* (meaning $g \wedge u = 0$ implies $g = 0$ for all $g \in G$), and whose morphisms are the ℓ-homomorphism preserving the weak unit.

Theorem. *The following are equivalent for* $(G, u) \in$ **W**.
  (a) (G, u) *is* **W** *epicomplete.*
  (b) (G, u) *is* **W** *isomorphic to* $(D(Y), 1)$.
  (c) (G, u) *is conditionally and laterally* σ-*complete, and* G *is divisible.*

1. INTRODUCTION

An epicomplete object G in a category is one for which the only morphisms out of G which are simultaneously epi and mono are the isomorphisms. We characterize the epicomplete objects in **Arch**, the category of archimedean lattice-ordered groups (archimedean ℓ-groups) with maps which preserve both group and lattice operations (ℓ-homomorphisms). We analyze **Arch** by working first in the closely related category **W**, whose objects are of the form (G, u), where G is an archimedean ℓ-group with weak unit u (meaning $0 < u \in G$ and

$u^{\perp} = \{g \in G: |g| \wedge u = 0\} = \{0\})$, and whose morphisms are ℓ-homomorphisms take the weak unit of the domain to the weak unit of the codomain.

Anderson and Conrad characterize epimorphisms in the category of abelian ℓ-groups in [AC] and then use the characterization to describe the epicomplete objects in that category. The identification of epimorphisms **Arch** and **W** turns out to be considerably more delicate and is carried out [BH I]. Madden and Vermeer use locales in [MV] to show that the epicomplete objects form a monoreflective subcategory of **W** (a result we re-prove here) but do so without explicitly characterizing the epicomplete objects. To supply this characterization, both in **W** and in **Arch**, is the point of this paper. We remark in passing that [BH III] is a rather complete analysis of this reflection from the point of view of groups of Baire measurable functions, though it is logically independent of all else. The present paper is self-contained, save only for its use of three fundamental theorems from [BH I], which we state in §2. Finally, it is a pleasure for the authors to acknowledge their gratitude to Professor D. G. Johnson, whose careful attention to preliminary versions of these papers improved them.

An archetypal **W** object is $(C(Y), 1)$, where $C(Y)$ designates the ℓ-group of continuous real-valued functions on the space Y and 1 is the constantly 1 function. (We assume without further comment the basic notation of [GJ], the classic reference on $C(Y)$.) To capture the full range of behaviors that arise in **W** objects, however, it is necessary to allow the elements to take on the values $\pm\infty$ occasionally, as follows. The *extended real line* $[-\infty, +\infty]$ is the two-point compactification of the real numbers obtained by adjoining points at $\pm\infty$, and a continuous extended real-valued function g on a space Y is *almost finite* provided that $g^{\leftarrow}(\mathbf{R}) = \{y \in Y: g(y) \in \mathbf{R}\}$ is dense in Y. $D(Y)$ designates the set of such functions; it is a lattice under pointwise supremum and infimum operations but may or may not be closed under the following addition operation. Given f, g, and h from $D(Y)$ we say $f + g = h$ provided that $f(y) + g(y) = h(y)$ for all $y \in f^{\leftarrow}(\mathbf{R}) \cap g^{\leftarrow}(\mathbf{R}) \cap h^{\leftarrow}(\mathbf{R})$. A **W** *object in* $D(Y)$ is a subset of $D(Y)$ which is closed under the aforementioned group and lattice operations and which contains the constantly 1 function as designated weak unit. Explicit mention of the weak unit is conventionally suppressed whenever no ambiguity results. The assertion that every **W** object is **W** isomorphic to a **W** object in some $D(Y)$ is the Yosida representation, outlined in §2. The best general reference to ℓ-groups is [BKW], and for the Yosida representation in particular [HR] is also very helpful.

If Y is the Stone space of a Boolean σ-algebra (i.e., a Boolean algebra whose every countable subset has both supremum and infimum), then $D(Y)$ is a **W** object in itself (we discuss this in more detail in §2), and we designate by **B** the full subcategory of **W** consisting of such objects. (The letter **B** honors Baire, since these objects are characterized in [BH III] as groups of Baire measurable

functions modulo null functions.) We remind the reader that a monomorphism $\iota: G \to H$ is extremal provided that $\iota = \tau\nu$ and ν epic imply ν is an isomorphism; in this case we term G an *extremal subobject* of H. Finally, an ℓ-group G is said to be conditionally (laterally) σ-complete provided that every countable bounded (pairwise disjoint) subset of positive elements of G has a supremum in G. We can now state the main results.

Theorem 3.9. *The following are equivalent for $G \in \mathbf{W}$.*

   (a) *G is $\mathbf{W}$-epicomplete.*

   (b) *$G \in \mathbf{B}$.*

   (c) *G is conditionally and laterally σ-complete and contains arbitrarily small positive multiples of its unit.*

Theorem 4.9. *The following are equivalent for $G \in \mathrm{Arch}$.*

   (a) *G is $\mathbf{Arch}$ epicomplete.*

   (b) *G is an $\mathbf{Arch}$ extremal subobject of a $\mathbf{B}$ object.*

   (c) *G is conditionally and laterally σ-complete, and G is divisible.*

2. The Yosida representation and fundamental theorems

The letters Y and X will be reserved for topological spaces, *which will henceforth be assumed to be compact Hausdorff.*

The Yosida representation of W objects. *For any $G \in \mathbf{W}$ there are a space $Y(G)$ and a $\mathbf{W}$ isomorphism ψ from G onto a $\mathbf{W}$ object in $D(Y(G))$ such that $\psi(G)$ separates the points of $Y(G)$. $Y(G)$ and ψ are unique in the sense that if X is another space and κ another $\mathbf{W}$ isomorphism such as ψ then there is a homeomorphism $\tau: X \to Y(G)$ such that $\kappa(g) = \psi(g)\tau$ for all $g \in G$.*

The space $Y(G)$ is called the *Yosida space of G*, and $\psi(G)$ is called the *Yosida representation of G*. We suppress explicit mention of ψ, writing $\hat{g}$ for $\psi(g)$. $Y(G)$ can be realized as the set of values of the unit u (i.e., convex ℓ-subgroups of G maximal with respect to omitting u) with the hull kernel topology; furthermore,

$$\psi(g)(P) = \begin{cases} +\infty & \text{if } g \notin Q \text{ and } P + g > P, \\ \theta(P + g) & \text{if } g \in Q, \\ -\infty & \text{if } g \notin Q \text{ and } P + g < P, \end{cases}$$

where P is a value of u, Q is the convex ℓ-subgroup generated by $G \cup \{u\}$, and θ is the unique ℓ-homomorphism from Q/P into $\mathbf{R}$ which takes u to 1.

A consequence of the uniqueness of the Yosida representation is the following useful fact. Given $(G, u) \in \mathbf{W}$ let G^* denote $\{g \in G: |g| \leq nu$ for some integer $n\}$, the convex ℓ-subgroup of bounded elements of G. Then $Y(G^*)$ is $Y(G)$, and the representing Yosida map on G restricts to that on G^*.

The Yosida representation is functorial in the following sense [HR].

The Yosida representation of W morphisms. *For any $\mathbf{W}$ morphism $\theta: G \to H$ there is a continuous map $\tau: Y(H) \to Y(G)$ which realizes θ in the sense that*

$\theta(g)^\wedge = \hat{g}\tau$. θ *is injective if and only if* τ *is surjective, and* τ *is injective whenever* θ *is surjective.*

If, as above, the elements of the Yosida spaces are realized as the values of the weak units, then $\tau(p)$ is $\theta^{\leftarrow}(p)$ for all $p \in Y(H)$.

For our purposes it is important to understand that a continuous map $\tau: Y(H) \to Y(G)$ realizes a **W** morphism $\theta: G \to H$ in the above sense if and only if $\tau^{\leftarrow}(\hat{g}^{\leftarrow}(\mathbf{R}))$ is dense in $Y(H)$ for each $g \in G$.

A *cozero set* $U \subset Y$ is one of the form $\text{coz}(g) = \{y \in Y : g(y) \neq 0\}$ for some $g \in D(Y)$; note that $f = 1/g \in D(Y)$ if and only if cc (g) is dense, in which case $\text{coz}(g) = f^{\leftarrow}(\mathbf{R})$. An arbitrary subset V is said to be C^*-*embedded in* Y provided each bounded continuous function on V has a continuous extension to Y. Y is said to be an *F-space* if each cozero subset is C^*-embedded in Y, and Y is said to be a *quasi F-space* if each dense cozero subset is C^* embedded in Y. F-spaces are discussed in 14.25–14.28 of [GJ], while [HVW] is an excellent source of information on quasi F-spaces, their literature, and their covering properties. The importance of quasi F-spaces derives from the fact that $D(Y)$ is a **W** object in itself (i.e., $D(Y)$ is closed under addition) if and only if Y is quasi F [HJ]. Y is *basically disconnected* provided that the closure of any cozero set is open (see 1H of [GJ]); such a space must have an open base of clopen (closed and open) subsets (see 16O of [GJ]) and so may be identified with the Stone space of $\text{clop}(Y)$, the Boolean algebra of clopen subsets of Y. Cozero sets in Boolean spaces (i.e., Stone spaces of Boolean algebras) are simply countable unions of clopen sets, from which one readily deduces that Y is basically disconnected if and only if Y is the Stone space of a Boolean σ-algebra. Finally, basically disconnected spaces can readily be shown to be F-spaces (see 14N4 of [GJ]). Thus we see that the objects of the full subcategory **B** of **W** are those of the form $D(Y)$ for basically disconnected spaces Y.

We turn now to the fundamental theorems in [BH I] characterizing epimorphisms in **W** and in **Arch**. Actually, these theorems characterize epi injections, since any morphism in these categories factors into a surjection followed by an injection, and the morphism will be epi when and only when the injection is epi. Suppose $G \leq H \in \mathbf{W}$, by which we mean that G and H are **W** objects and the inclusion map is a **W** morphism, and suppose the embedding is realized by $\tau: Y(H) \to Y(G)$. For $0 < h \in H$ let

$$\Sigma(h) = \{(p, q) \in Y(H)^2 : \tau(p) = \tau(q) \text{ and } \hat{h}(p) \neq \hat{h}(q)\}.$$

The evaluations are carried out in the Yosida representation of H.

The Fundamental Theorem for W. $G \leq H \in \mathbf{W}$ *is* **W** *epi if and only if for each* $0 < h \in H$ *there is a countable set* $K \subset H$ *satisfying the following condition. For every* $(p, q) \in \Sigma(h)$ *there is some* $k \in K$ *such that either* $\hat{k}(p) = \infty$ *or* $\hat{k}(q) = \infty$.

The elements of K are called *epi indicators for h*. When verifying the hypotheses of the fundamental theorem is is enough to produce epi indicators for a generating subset of H.

Proposition. *Suppose that $G \leq M \in \mathbf{W}$, that $L \subset M$, and that H is the ℓ-subgroup of M generated by $G \cup L$. If every element of ℓ has a countable set of epi indicators in H then $G \leq H$ is $\mathbf{W}$ epi.*

Proof. Each element $h \in H$ is a finitary combination of members of $G \cup L$, and the countably many epi indicators for these members will serve for h also. $\square$

The analysis of epimorphisms in **Arch** can be reduced to the consideration of those in **W** by the following strategem. Let $G \leq H \in \mathbf{Arch}$, fix $0 < u \in G$, let $u^{\perp G}$ ($u^{\perp H}$) designate the polar $\{x \in G(H): |x| \wedge u = 0\}$, and let $\pi_G: G \to G/u^{\perp G}$ and $\pi_H: H \to H/u^{\perp H}$ be the natural **Arch** morphisms. Because $u^{\perp H} \cap G = u^{\perp G}$, there is a unique **Arch** morphism $\delta: G/u^{\perp G} \to H/u^{\perp H}$ such that $\pi_H|_G = \delta \pi_G$.

$$
\begin{array}{ccc}
G & \leq & H \\
\pi_G \downarrow & & \pi_H \downarrow \\
G/u^{\perp G} & \xrightarrow{\delta} & H/u^{\perp H}
\end{array}
$$

Two features of δ are crucial: first that $\delta: (G/u^{\perp G}, \pi_G(u)) \to (H/u^{\perp H}, \pi_H(u))$ is actually a **W** injection, and second that δ is **W** epi whenever $G \leq H$ is **Arch** epi. (Proof: if $\alpha_1, \alpha_2 \in \mathbf{W}$ satisfy $\alpha_1 \delta = \alpha_2 \delta$ then also $\alpha_1 \pi_H|_G = \alpha_1 \delta \pi_G = \alpha_2 \delta \pi_G = \alpha_2 \pi_H|_G$ in **Arch**, so $\alpha_1 = \alpha_2$).

A second ingredient is necessary for the characterization of epimorphisms in **Arch**. We shall term $G \leq H \in \mathbf{Arch}$ *coessential* provided the only **Arch** morphism on H which takes each element of G to 0 is the zero morphism.

The Fundamental Theorem for Arch. $G \leq H \in \mathbf{Arch}$ *is* **Arch** *epi if and only if it is coessential and for every $0 < u \in G$ the map $\delta: G/u^{\perp G} \to H/u^{\perp H}$ is* **W** *epi.*

[BH I] contains a thorough discussion of these theorems and several limiting examples showing the indispensability of each of the hypotheses mentioned.

A drawback of the Fundamental Theorem for **Arch** is the difficulty of verifying that $\delta: G/u^{\perp G} \to H/u^{\perp H}$ is a **W** epimorphism for *each* choice of $0 < u \in G$. These verifications may be replaced by a single one, and the proof of coessentiality omitted, in case $G \leq D(Y) \in \mathbf{W}$ for some quasi-F space Y. This result, which is a crucial ingredient of the proof of Lemma 4.1, is a consequence of the ring structure of $D(Y)$. Details and proof may be found in §5 of [BH I].

The Ring Theorem. *Suppose that $G \leq D(Y) \in \mathbf{W}$ for some quasi-F space Y. If the embedding is* **W** *epi then it is* **Arch** *epi.*

3. Epicompleteness in W

In this section we show that the epicomplete objects in **W** are precisely the **B** objects (Theorem 3.9). We begin by remarking that the computations for epi

indication in the Fundamental Theorem for **W** may be carried out in any **W** supergroup. In particular, let $G \leq H \leq M \in \mathbf{W}$ be realized by $\sigma: Y(M) \rightarrow Y(H)$ and $\rho: Y(H) \rightarrow Y(G)$, and let $\tau = \rho\sigma$. Fix $0 < h \in H$, and let $\hat{h}$ refer to the Yosida representation of H. Let

$$\Sigma_G^H(h) = \{(p, q) \in Y(H)^2 : \rho(p) = \rho(q) \text{ and } \hat{h}(p) \neq \hat{h}(q)\},$$

and let

$$\Sigma_G^M(h) = \{(p', q') \in Y(M)^2 : \tau(p') = \tau(q') \text{ and } \hat{h}\sigma(p') \neq \hat{h}\sigma(q')\}.$$

Lemma 3.1. *Suppose* $G \leq H \leq M \in \mathbf{W}$. *Then* $G \leq H$ *is* **W** *epi if and only if for every* $0 < h \in H$ *there is some countable set* $K \subset H$ *satisfying the following condition. For every* $(p', q') \in \Sigma_G^M(h)$ *there is some* $k \in K$ *such that either* $\hat{k}\sigma(p') = \infty$ *or* $\hat{k}\sigma(q') = \infty$.

Proof. If $(p', q') \in \Sigma_G^M(h)$ then by definition $(\sigma(p'), \sigma(q')) \in \Sigma_G^H(h)$, so that the set K above consists of epi indicators for h in the usual sense of the Fundamental Theorem for **W**. On the other hand, given $p, q \in Y(H)$ there exist $p', q' \in Y(M)$ such that $\sigma(p') = p$ and $\sigma(q') = q$, and $(p', q') \in \Sigma_G^M$ if $(p, q) \in \Sigma_G^H$. Therefore any set K of epi indicators in the usual sense satisfies the condition above. □

Lemma 3.2. *Suppose that* X *is a basically disconnected space and that the* **W** *embedding* $G \leq D(X)$ *is realized by* $\tau: X \rightarrow Y(G)$.

(a) *For every cozero* $U \subset Y(G)$ *there is an* ℓ-*subgroup* $H \leq D(X)$ *containing* G *together with the characteristic function* χ *of* $\mathrm{cl}(\tau^{\leftarrow}(U))$ *such that* $G \leq H$ *is* **W** *epic.*

(b) *For every* $f \in D(Y(G))$ *there is an* $h \in D(X)$ *such that* $h(x) = f\tau(x)$ *whenever* $\tau(x) \in f^{\leftarrow}(\mathbf{R})$ *and such that* $G \leq H$ *is* **W** *epi, where* H *is the* ℓ-*subgroup of* $D(X)$ *generated by* $G \cup \{h\}$.

Proof. (a) Let U be the cozero set of $v \in D(Y(G))$, and assume $0 \leq v \leq u$, where u is the designated weak unit of G. Since $\tau^{\leftarrow}(U)$ is a cozero set in X, $\mathrm{cl}(\tau^{\leftarrow}(U))$ is open in X and $\chi \in D(X)$. Define

$$h(x) = \begin{cases} 1/v\tau(x) & \text{for } x \in \tau^{\leftarrow}(U), \\ +\infty & \text{for } x \in \mathrm{cl}(\tau^{\leftarrow}(U)) - \tau^{\leftarrow}(U), \\ 0 & \text{for } x \notin \mathrm{cl}(\tau^{\leftarrow}(U)). \end{cases}$$

A little reflection reveals that h lies in $D(X)$. We apply Lemma 3.1 to this situation, with M taken to be $D(X)$ and H taken to be the ℓ-subgroup generated by $G \cup \{\chi, h\}$. Given $(x, y) \in \Sigma_G^M(\chi)$, we see that either x or y must lie in $\mathrm{cl}(\tau^{\leftarrow}(U)) - \tau^{\leftarrow}(U)$, and thus that h epi indicates for χ. Likewise, h epi indicates for itself. We conclude that $G \leq H$ is **W** epi.

(b) Given $0 < f \in D(Y(G))$ let U be $\tau^{\leftarrow}(f^{\leftarrow}(\mathbf{R}))$, a cozero subset of X. Define

$$h(x) = \begin{cases} f\tau(x) & \text{for } x \in U, \\ +\infty & \text{for } x \in \mathrm{cl}(U) - U, \\ 0 & \text{for } x \notin \mathrm{cl}(U). \end{cases}$$

Keeping in mind that $\mathrm{cl}(U)$ is open, one readily verifies that $h \in D(X)$. If M and H are taken to be $D(X)$ and the ℓ-subgroup generated by $G \cup \{h\}$, respectively, then we see that $(x, y) \in \Sigma_G^M(h)$ implies that either x or y lies in $\mathrm{cl}(U) - U$. Lemma 3.1 then asserts that $G \leq H$ is **W** epi. □

We use V^c to designate the set theoretical complement of the subset $V \subset Y$.

Proposition 3.3. *A* **W** *extremal subobject of a* **B** *object is itself a* **B** *object.*

Proof. Suppose X is a basically disconnected space, and suppose that G is a **W** extremal subobject of $D(X)$ with $\tau: X \to Y(G)$ realizing $G \leq D(X)$. Consider an arbitrary cozero subset $U \subset Y(G)$, and let χ be the characteristic function of $V = \mathrm{cl}(\tau^{\leftarrow}(U)) \subset X$. Now G admits no epi enlargements in $D(X)$, so Lemma 3.2(a) yields $\chi \in G$, and this implies that $\tau(V) \cap \tau(V^c) = \Phi$, meaning $\tau(V)$ is clopen. But since $U \subset \tau(V) \subset \mathrm{cl}(U)$, we have $\mathrm{cl}(U) = \tau(V)$ open. That is, $Y(G)$ is basically disconnected.

Now consider $f \in D(Y(G))$ and produce $h \in D(X)$ as in Lemma 3.2(b). As before, the extremality of G in $D(X)$ with Lemma 3.2(b) yields $h \in G$. By abusing the notation slightly, we may regard $\hat{h}$ to lie in $D(Y(G))$, from which vantage we see that h and f agree on the dense set $f^{\leftarrow}(\mathbf{R})$ and so coincide. This proves that $G = D(Y(G)) \in \mathbf{B}$. □

An ℓ-subgroup G is *large* in the ℓ-group H if every nontrivial convex ℓ-subgroup of H intersects G nontrivially. This is evidently equivalent to the essentiality of the extension in the category of abelian ℓ-groups: any homomorphism on H which is one-one on G is one-one on H as well. Bernau showed in [B] that for any **W** object G there is a unique **B** object B and a unique W embedding $G \leq B$ such that G is large in B. Furthermore, Conrad showed in [C1] that G is large in a **W** extension H if and only if there is **W** monomorphism from H into B over G.

Theorem 3.4. *If* G *lacks proper* **W** *epi extensions in which it is large, then* G *is a* **B** *object; in particular, if* G *is* **W** *epicomplete then it is a* **B** *object.*

Proof. Let $G \leq B$ be the above-mentioned **W** embedding of G into the **B** object B in which G is large. Let K be the ℓ-subgroup of $D(X)$ generated by $\bigcup \mathcal{K}$, where $\mathcal{K}$ is $\{L \leq B: G \leq L$ is **W** epi$\}$. Then $G \leq K$ is **W** epic because any pair of **W** morphisms with common codomain which agree on G must agree on each member of $\mathcal{K}$ and so also on K. Therefore G has no proper **W** epi extensions in which it is large if and only if $G = K$, and K is a **B** object by Theorem 3.3. □

Theorem 3.7 depends on a subtlety of the Yosida representation made explicit in the following lemma. **Boolσ** designates the category whose objects are the Boolean σ-algebras (i.e., Boolean algebras whose every countable subset has both supremum and infimum) and Boolean σ-homomorphisms (i.e., Boolean homomorphisms which preserve all countable suprema and infima).

Lemma 3.5. *Suppose* $\tau: Y \to X$ *is a continuous map between basically discon-nected spaces, and let* $\tau^{\leftarrow}: \mathrm{clop}(X) \to \mathrm{clop}(Y)$ *be the Boolean homomorphism given by Stone duality. Then* τ *realizes a* **W** *morphism* $\theta: D(X) \to D(Y)$ *if and only if* $\tau^{\leftarrow}$ *is a Boolean σ-homomorphism.*

Proof. τ realizes θ if and only if $\tau^{\leftarrow}(U)$ is a dense cozero of Y whenever U is a dense cozero of X. (This condition is necessary because each such U is of the form $g^{\leftarrow}(\mathbf{R})$ for some $g \in D(X)$, and since $g\tau = \theta(G)$ must lie in $D(Y)$, $\theta(g)^{\leftarrow}(\mathbf{R}) = \tau^{\leftarrow}(U)$ must be dense in Y. The condition is sufficient since in its presence $\theta(g) = g\tau$ lies in $D(Y)$ for $g \in D(X)$.) When we recall that the cozero subsets of a Boolean space are the countable unions of clopen sets and that such a union $\bigcup_N C_n$ is dense precisely when $\bigvee_N C_n = X$ in $\mathrm{clop}(X)$, we see that the crucial issue is whether or not $\tau^{\leftarrow}$, viewed as a Boolean homomorphism, preserves this countable supremum. But preservation of countable suprema of this type is well known to be equivalent to preservation of all countable suprema and infima. $\square$

Corollary 3.6. **B** *objects are* **W** *epicomplete if and only if* **Boolσ** *epimorphisms are surjective.*

Proof. Consider a **B** object B and **W** epi extension $B \le H$. Let $H \le E$ be the **W** epi embedding of H into the **B** object E discussed in the proof of Theorem 3.4; then $B \le E$ is **W** epi. Identify B and E with $D(Y(B))$ and $D(Y(E))$, respectively, let $\tau: Y(E) \to Y(B)$ designate the surjection realizing the embedding of B in E, and consider $\tau^{\leftarrow}: \mathrm{clop}(Y(B)) \to \mathrm{clop}(Y(E))$. Now $\tau^{\leftarrow}$ is a **Boolσ** morphism by Lemma 3.5; we claim that $\tau^{\leftarrow}$ is also a **Boolσ** epimorphism. For if $\rho_1, \rho_2: \mathrm{clop}(Y(E)) \to T$ are **Boolσ** morphisms such that $\rho_1 \tau^{\leftarrow} = \rho_2 \tau^{\leftarrow}$, then without loss of generality we may take T to be of the form $\mathrm{clop}(Z)$ for some basically disconnected space Z. By Lemma 3.5 we get **W** morphisms $\theta_1, \theta_2: E \to D(Z)$ such that θ_1 and θ_2 agree on B, whereupon the epicity of $B \le E$ forces $\theta_1 = \theta_2$, which in turn forces $\rho_1 = \rho_2$. This proves the claim that $\tau^{\leftarrow}$ is epi in **Boolσ**. If epimorphisms in **Boolσ** are surjective then τ is a homeomorphism and $B = E$, which forces $B = H$ and shows B to be epicomplete in **W**. On the other hand, Stone duality asserts that an arbitrary **Boolσ** morphism may be viewed as having the form $\tau^{\leftarrow}: \mathrm{clop}(X) \to \mathrm{clop}(Y)$ for a continuous map $\tau: Y \to X$ between basically disconnected spaces X and Y, and Lemma 3.5 asserts that this τ realizes a **W** morphism $\theta: D(X) \to D(Y)$. Therefore the argument above can be read in reverse to prove the converse. $\square$

Lagrange proved in [L] that **Boolσ** epimorphisms are surjective, which es-tablishes the following theorem. We give here a quite different proof which is self-contained and, of course, proves Lagrange's theorem.

Theorem 3.7. **B** *objects are* **W** *epicomplete.*

Proof. Let B, H, E, and τ have their meaning in the proof of the previous corollary. We shall prove that $\tau(C) \cap \tau(C^c) = \varnothing$ for any $C \in \mathrm{clop}(Y(E))$, and this will show the injectivity of τ because $Y(E)$ has a clopen basis. Fix

$C \in \operatorname{clop}(Y(E))$, let $\chi \in E$ be its characteristic function, and then use the Fundamental Theorem for **W** to find epi indicators $\{e_n : n \in N\} \subset E$ for χ. We may assume without loss of generality that $0 = e_1 \leq e_2 \leq \cdots$. The definition of epi indication translates in this context to the condition that $\tau(F \cap C) \cap \tau(F \cap C^c) = \varnothing$, where F designates $\bigcap_N e_n^{\leftarrow}(\mathbf{R})$. Let

$$\mathscr{T} = \{K \in \operatorname{clop}(Y(E)): \tau(K \cap C) \cap \tau(K \cap C^c) = \varnothing\},$$

and let

$$\mathscr{K}_n = \{K \in \operatorname{clop}(Y(E)): K \subset e_n^{\leftarrow}(\mathbf{R})\}.$$

Since $Y = e_1^{\leftarrow}(\mathbf{R}) \in \mathscr{K}_1$, we will prove the theorem by showing in three claims that $\mathscr{K}_1 \subset \mathscr{T}$.

Claim 1. Either $\mathscr{K}_1 \subset \mathscr{T}$ or there are a clopen set K and an index n for which $K \in \mathscr{K}_n - \mathscr{T}$ but $\{L \in \mathscr{K}_{n+1} : \varnothing \neq L \subset K\} \subset \mathscr{T}$.

Proof of Claim 1. Suppose the claim fails. Choose $K_1 \in \mathscr{K}_1 - \mathscr{T}$, and for $n > 1$ choose $K_{n+1} \in \mathscr{K}_{n+1} - \mathscr{T}$ such that $\varnothing \neq K_{n+1} \subset K_n$. Since $\tau(K_n \cap C) \cap \tau(K_n \cap C^c) \neq \varnothing$ for each n, there exists $x \in \bigcap_N (\tau(K_n \cap C) \cap \tau(K_n \cap C^c))$. Thus $\tau^{\leftarrow}(x) \cap K_n \cap C \neq \varnothing$ for all n, so there is some $y_1 \in \tau^{\leftarrow}(x) \cap (\bigcap_N K_n) \cap C$ and likewise some $y_2 \in \tau^{\leftarrow}(x) \cap (\bigcap_N K_n) \cap C^c$. But since $\tau(y_1) = \tau(y_2) = x$ and $\bigcap_N K_n \subset F$, we have contradicted the assumption that $\tau(F \cap C) \cap \tau(F \cap C^c) = \varnothing$ and proved the claim.

Claim 2. $\mathscr{T} = \{K \in \operatorname{clop}(Y(E)): K \cap \tau^{\leftarrow}(A) = K \cap C \text{ for some } A \in \operatorname{clop}(Y(B))\}$.

Proof of Claim 2. If $K \cap \tau^{\leftarrow}(A) = K \cap C$ for some $A \in \operatorname{clop}(Y(B))$ then $\tau(K \cap C) \subset A$ and $\tau(K \cap C^c) \subset A^c$, meaning $K \in \mathscr{T}$. On the other hand, $\tau(K \cap C) \cap \tau(K \cap C^c) = \varnothing$ implies the existence of a clopen set A such that $\tau(K \cap C) \subset A$ and $\tau(K \cap C^c) \subset A^c$, so $K \cap C = K \cap \tau^{\leftarrow}(A)$.

Claim 3. $\mathscr{K}_1 \subset \mathscr{T}$.

Proof of Claim 3. Assuming the contrary, find $K \in \mathscr{K}_n - \mathscr{T}$ for which $\{L \in \mathscr{K}_{n+1} : \varnothing \neq L \subset K\} \subset \mathscr{T}$. Because $Y(E)$ is basically disconnected, $\operatorname{cl}(e_{n+1}^{\leftarrow}(0, i))$ is clopen for each positive integer i; define L_i to be $K \cap \operatorname{cl}(e_{n+1}^{\leftarrow}(0, i))$. By Claim 2 find $A_i \in \operatorname{clop}(X)$ for each i such that $L_i \cap C = L_i \cap \tau^{\leftarrow}(A_i)$. For each positive integer m we label $\bigwedge_{i \geq m} A_i$ by A'_m, an element whose existence is assured by the σ-completeness of $\operatorname{clop}(Y(B))$. We need the following two facts concerning A'_m.

(1) $L_m \cap C \subset L_m \cap \tau^{\leftarrow}(A'_m)$. To see this observe that $m \leq i$ implies $L_m \cap C \subset L_i \cap C = L_i \cap \tau^{\leftarrow}(A_i)$, so $L_m \cap C \subset L_m \cap \tau^{\leftarrow}(A'_m)$ because $\tau^{\leftarrow}$ preserves countable infima.

(2) $K \cap \tau^{\leftarrow}(A'_m) \subset K \cap C$. This is so because $\bigvee_{i \geq m} L_i = K$ and $L_i \cap C^c = L_i \cap \tau^{\leftarrow}(A_i)^c$ imply

$$C^c \cap K = \bigvee_{i \geq m} (L_i \cap C^c) = \bigvee_{i \geq m} (L_i \cap \tau^{\leftarrow}(A_i)^c) \leq \bigvee_{i,j \geq m} (L_i \cap \tau^{\leftarrow}(A_j)^c)$$

$$= \left(\bigvee_{i \geq m} L_i \right) \cap \left(\bigvee_{j \geq m} \tau^{\leftarrow}(A_j)^c \right) = K \cap \tau^{\leftarrow}(A'_m)^c,$$

which implies $C^c \cap K \cap \tau^{\leftarrow}(A'_m) = \varnothing$, hence $K \cap \tau^{\leftarrow}(A'_m) \subseteq C$, which is what we wanted. Now set $A = \bigvee_N A'_m \in \text{clop}(Y(B))$, and apply (1) to get the first containment and (2) the last in the following:

$$K \cap C = \bigvee_N (L_m \cap C) \subset \bigvee_N (K_m \cap \tau^{\leftarrow}(A'_m)) \subset \bigvee_N (K \cap \tau^{\leftarrow}(A'_m)) \subset K \cap C.$$

Since $\bigvee_N (K \cap \tau^{\leftarrow}(A'_m)) = K \cap \bigvee_N \tau^{\leftarrow}(A'_m) = K \cap \tau^{\leftarrow}(A)$, we have contradicted the assumption that $K \notin \mathcal{T}$, thereby proving the claim and theorem. $\square$

The Stone-Nakano theorem asserts that $C(Y)$ is conditionally (σ) complete if and only if Y is extremally (basically) disconnected [S, GJ]. The following lemma is the version of this celebrated theorem best suited to proving the Main Theorem 3.9 of this section. The curious requirement (a) of this lemma is necessary to prevent counterexamples of the form $\{g \in C(Y): g$ is integer valued$\}$ for Boolean spaces Y. We are indebted to Professor D. G. Johnson for simplifying the argument.

Lemma 3.8. *A* **W** *object (G, u) has properties* (a) *and* (b) *below if and only if it has properties* (c) *and* (d).
    (a) *G contains arbitrarily small positive multiples of u.*
    (b) *G^* is conditionally σ-complete.*
    (c) *$Y(G)$ is basically disconnected.*
    (d) *The Yosida representation carries G^* onto $C(Y(G))$.*

Proof. (d) clearly implies (a); if (c) and (d) hold then, since $Y(G^*) = Y(G)$ and the Yosida representation of G restricts to the Yosida representation of G^*, the Stone-Nakano theorem implies (b). Now assume G enjoys properties (a) and (b); we aim to show G satifies (d). We adopt the viewpoint that G is a **W** object in $D(Y(G))$; i.e., we identify G with its Yosida representation, and consider arbitrary $0 < h \in C(Y(G))$. We claim that for each positive real number ε such that $\varepsilon u \in G$ there is some $0 < g_\varepsilon \in G$ such that $h - \varepsilon u \leq g_\varepsilon \leq h$. To prove this claim, first find for each $x \in \text{coz}(h)$ a real number $r > 0$ such that $ru \in G$ and $h(x) - \varepsilon < r < h(x)$. Then choose $k_x \in G$ such that $\text{coz}(k_x) \subset \text{coz}((h - ru) \vee 0)$ and $k_x(x) > r$. Set $g_x = k_x \wedge ru$, and observe that $g_x \in G$, that $0 < g_x \leq h$, and that $h(x) - \varepsilon < g_x(x)$. Now if $V_x = \text{coz}((g_x + \varepsilon u - h) \vee 0)$ then $\{V_x : x \in \text{coz}(h)\}$ covers $\text{coz}(h)$ and therefore covers $\text{cl}(\text{coz}((h - \varepsilon u) \vee 0))$; let $\{V_{x_i} : 1 \leq i \leq n\}$ be a finite subcover. Finally, set $g_\varepsilon = \bigvee_1^n g_{x_i}$, and check that this element has the properties claimed for it.

Let E be a countable set of positive real numbers such that $\bigwedge E = 0$ and such that $\varepsilon u \in G$ for all $\varepsilon \in E$. (That such a set exists is a consequence of (a).) For each $\varepsilon \in E$ find $g_\varepsilon \in G$ as above. Since h is bounded by some multiple of u, $\bigvee_E g_\varepsilon = g$ exists in G. Since both G and $C(Y(G))$ are large in $D(Y(G))$, suprema agree in all three groups. We conclude that $h = g \in G$, thus proving (d). The Stone-Nakano theorem then yields (c). $\square$

Theorem 3.9. *The following are equivalent for* $G \in \mathbf{W}$.
  (a) G *is* $\mathbf{W}$ *epicomplete.*
  (b) $G \in \mathbf{B}$.
  (c) G *is conditionally and laterally* σ-*complete, and* G *contains arbitrarily small positive multiples of its unit.*

Proof. Theorem 3.4 proves that (a) implies (b). To show (b) implies (c) consider $G = D(Y)$ for some basically disconnected space Y. Such a G certainly contains arbitrarily small positive multiples of its unit. To show G laterally σ-complete consider the pairwise disjoint subset $\{g_n : n \in N\}$, and let g be the pointwise supremum of the g_n's. Now g is evidently continuous, and the open set $g^{\leftarrow}(\mathbf{R})$ contains the dense set $\bigcap_n g_n^{\leftarrow}(\mathbf{R})$ because of the disjointness of the g_n's. Therefore $\bigvee_N g_n = g \in G$. To show G conditionally σ-complete consider $\{g_n : n \in N\} \subset G$ such that $0 \leq g_n \leq g$ for some $g \in G$ and all n. Now G^* is conditionally σ-complete by the Stone-Nakano theorem, so $\bigvee_N(g_n \wedge ku) = f_k$ exists in G for each positive integer k, though this supremum need not be pointwise in general. However, the pointwise supremum f of the f_k's is readily seen to be continuous, and f lies in G by virtue of being dominated by g. Clearly $\bigvee_N g_n = f$.

Now suppose that G satisfies (c), and identify G with its Yosida representation. Then Lemma 3.8 shows $Y(G)$ to be basically disconnected and shows $G^* = C(Y(G))$. To prove $G = D(Y(G))$ consider $0 < h \in D(Y(G))$. For each nonnegative integer n let C_n be the clopen set $\mathrm{cl}(\mathrm{coz}(h - n) \vee 0)$, let D_n designate $C_n - C_{n+1}$, and let g_n designate that member of G which agrees with h on D_n and is 0 off D_n. Since the C_n's are nested the D_n's are disjoint, and so also are the g_n's. Let g be the supremum of the g_n's provided by the lateral σ-completeness of G. Now $\bigvee_N g_n = h$ in $D(Y(G))$, and suprema in G and $D(Y(G))$ agree because the former is large in the latter, so $h = g \in G$, proving (b). Finally, Theorem 3.7 proves that (b) implies (a). $\square$

In [BH III] we prove that $\mathbf{B}$ is closed under arbitary $\mathbf{W}$ surjections; a weaker version of this result is the following corollary, which will be used in §4. A convex ℓ-subgroup C of an ℓ-group is *closed* if $0 \leq S \subset C$ and $\bigvee S = g$ in G imply that $g \in C$. It is known [BKW] that the closed convex ℓ-subgroups of an archimedean ℓ-group are precisely the polars. Moreover, an ℓ-homomorphism preserves all suprema and infima if and only if its kernel is closed [BL].

Corollary 3.10. *If* $0 < u \in B \in \mathbf{B}$ *then* $(B/u^{\perp}, u^{\perp} + u) \in \mathbf{B}$.

Proof. Let $\pi: B \to B/u^\perp$ be the natural map. To show $B/u^\perp$ conditionally σ-complete consider $\{\pi(b_n): n \in N\}$ and $\pi(b)$ such that $0 \leq \pi(b_n) \leq \pi(b)$ for all n. Without loss of generali.y we may assume that $0 \leq b_n \leq b$ for all n. Then the conditional σ-completeness of B provides $c = \bigvee_N b_n$ in B, and the closure of $u^\perp$ gives $\pi(c) = \bigvee_N \pi(b_n)$ in $B/u^\perp$. The arguement that $B/u^\perp$ is laterally σ-complete is similar, and since $B/u^\perp$ is evidently divisible we conclude that $B/u^\perp \in \mathbf{B}$ by part (c) of Theorem 3.9. $\square$

By using locales, Madden and Vermeer proved in [MV] that the epicomplete objects form a monoreflective subcategory of $\mathbf{W}$, a result we are now in a position to confirm by an argument quite different from theirs.

Corollary 3.11. *For any* $G \in \mathbf{W}$ *there is a* $\mathbf{W}$ *epicomplete object* B_G *and* $\mathbf{W}$ *epic injection* $\iota_G: G \to B_G$ *with the universal mapping property: for any* $\mathbf{W}$ *epicomplete object* B *and* $\mathbf{W}$ *morphism* $\theta: G \to B$ *there is a unique* $\mathbf{W}$ *morphism* $\tau: B_G \to B$ *such that* $\theta = \tau\iota_G$.

Proof. Since $\mathbf{W}$ is co-well powered [BH I], it is sufficient to show that $\mathbf{B}$ is closed under products and extremal subobjects (see Chapter 10 of [HS]). The closure of $\mathbf{B}$ under extremal subobjects is Proposition 3.3, and one can readily check the persistence of lateral and conditional σ-completeness under the operation of taking products in $\mathbf{W}$. $\square$

4. Epicompleteness in Arch

We prove that the epicomplete objects in **Arch** are the **Arch** extremal subobjects of $\mathbf{B}$ objects (Theorem 4.9). The proof proceeds by reducing the situation in **Arch** to that in $\mathbf{W}$ by means of the construct mentioned in the introduction, which we reiterate here to fix the notation for the remainder of the section. Given an **Arch** injection $\iota: G \to H$ and $0 < u \in G$, $\pi_G: G \to G/u^\perp$ and $\pi_H: H \to H/\iota(u)^\perp$ denote the natural **Arch** morphisms, and $\delta: (G/u^\perp, \pi_G(u)) \to (H/\iota(u)^\perp, \pi_H\iota(u))$ is the unique $\mathbf{W}$ injection making the diagram commute:

$$\begin{array}{ccc} G & \overset{\iota}{\longrightarrow} & H \\ \pi_G\downarrow & & \pi_H\downarrow \\ G/u^\perp & \overset{\delta}{\longrightarrow} & H/\iota(u)^\perp. \end{array}$$

Then δ is $\mathbf{W}$ epi whenever ι is **Arch** epi.

We have occasion in the following propositon to mention the well-known principal projection property. For any $G \in \mathbf{Arch}$ and any $0 < u \in G$ we note that the natural **Arch** morphisms $\pi: G \to G/u^\perp$ and $\psi: G \to G/u^{\perp\perp}$ (here $u^{\perp\perp}$ designates $\bigcap\{g^\perp: g \in u^\perp\}$) jointly induce an **Arch** injection μ from G into the product $G/u^\perp \oplus G/u^{\perp\perp}$. If each such μ is onto then G is said to have *the principal projection property*, and in case G enjoys this property we refer to $\pi(g)$ $(\psi(g))$ as the *projection of g on* $u^{\perp\perp}$ $(u^\perp)$. Two remarks on this property are germane in what follows.

For a particular $0 < u \in G$, the formula for its μ is explicitly given by $\mu(g) = (\pi(g), \psi(g))$. If μ is onto then for each $g \in G$ there are $g_1, g_2 \in G$
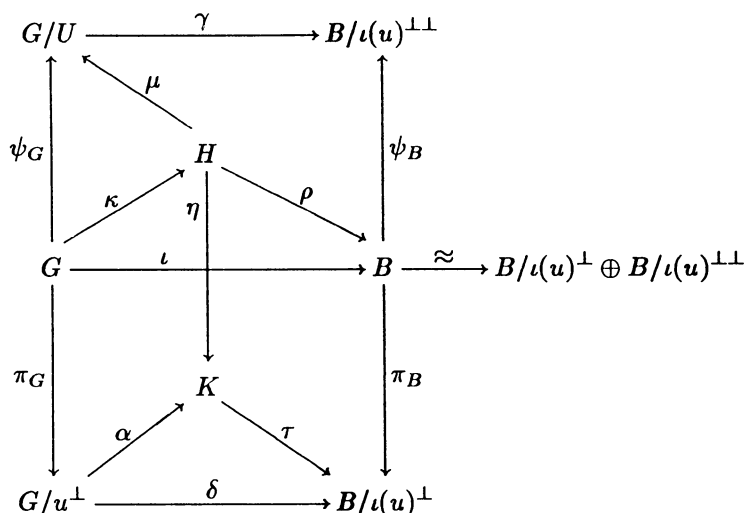
such that $\mu(g_1) = (\pi(g), 0)$ and $\mu(g_2) = (0, \psi(g))$. Now $\psi(g_1) = 0$ implies $g_1 \in \ker(\psi) = u^{\perp\perp}$, and similarly $g_2 \in u^{\perp}$. Furthermore, $g_1 + g_2 = g$ because $\mu(g_1 + g_2) = (\pi(g), \psi(g)) = \mu(g)$. Conversely, if for each $g \in G$ there exist $g_1, g_2 \in G$ such that $g_1 \in u^{\perp\perp}$, $g_2 \in u^{\perp}$, and $g_1 + g_2 = g$, then μ can readily be shown to be onto. Observe that in this case the maps $\pi(g) \to g_1$ and $\psi(g) \mapsto g_2$ are **Arch** isomorphisms from $G/u^{\perp}$ onto $u^{\perp\perp}$ and from $G/u^{\perp\perp}$ onto $u^{\perp}$, respectively.

The second remark is that a sufficient condition for the possession of the principal projection property is conditional σ-completeness. (Proof: for $0 < g \in G$ let $\pi(g) = \bigvee_N (nu \wedge g)$ and $\psi(g) = g - \pi(g)$.)

Proposition 4.1. *Suppose $\iota: G \to B$ is an* **Arch** *monomorphism, suppose that $B \in \mathbf{B}$, fix $0 < u \in G$, and let π_G, π_B, and δ have the meaning above. Then δ is* **W** *extremal whenever ι is* **Arch** *extremal.*

Proof. Let $U = \{g \in G : \iota(g) \in \iota(u)^{\perp\perp}\}$, let $\psi_G: G \to G/U$ and $\psi_B: B \to B/\iota(u)^{\perp\perp}$ be the natural **Arch** morphisms, and let γ be the induced **Arch** injection which makes the following diagram compute.

To check that δ is **W** extremal, consider the factorization $\delta = \tau\alpha$, where $\alpha: G/u^{\perp} \to K$ is a **W** epimorphism. Now $B/\iota(u)^{\perp} \in \mathbf{B}$ by Corollary 3.10, and by replacing K by its maximal epic enlargement in the latter we may assume that $K \in \mathbf{B}$ by Proposition 3.3. Since K is of the form $D(X)$ for some basically disconnected compact Hausdorff space X, the Ring Theorem of §1 shows that α is actually **Arch** epi.

We can lift α and τ to corresponding **Arch** injections κ and ρ as follows. Let H designate the cardinal product of K with G/U, with projection maps $\eta: H \to K$ and $\mu: H \to G/U$. Now $\gamma\mu$ and $\tau\eta$ jointly produce an

Arch injection from H into $B/\iota(u)^{\perp\perp} \oplus B/\iota(u)^{\perp}$, and this product is isomorphic to B because B has the principal projection property as a result of its σ-completeness. Let $\rho: H \to B$ be the **Arch** mapping so induced. Likewise ψ_G and $\alpha\pi_G$ jointly produce an **Arch** morphism $\kappa: G \to H$ such that $\rho\kappa = \iota$. We shall show that κ is **Arch** epi, whereupon the **Arch** extremal property of ι will force κ to be surjective, which in turn will force α to be surjective and thus show δ to be a **W** extremal monomorphism.

We must verify that κ satisfies the hypotheses of the Fundamental Theorem for **Arch** of §1. For that purpose let us agree to represent each $h \in H$ in the form $h_1 + h_2$, where $h_1 \in K$, $h_2 \in G/U$, and $|h_1| \wedge |h_2| = 0$. Let us further agree to identify each $g \in G$ with $\kappa(g)$, so that $g_1 = \alpha\pi_G(g)$ and $g_2 = \psi_G(g)$. To show that G is coessential in H, consider an **Arch** morphism θ out of H such that $\theta(G) = 0$; we need to show that an arbitrarily chosen $h \in H$ satisfies $\theta(h) = 0$. Part of this task is easy: since ψ_G is onto there is some $g \in G^+$ such that $g_2 = h_2$, and since $0 \leq h_2 = g_2 \leq g$, we see that $\theta(h_2) = 0$. The other part is not much harder. Since K is a cardinal summand of H, for each $g \in G^+$ we have $\theta\alpha\pi_G(g) = \theta\eta\kappa(g) = \theta(g_1) \leq \theta(g) = 0$. And since $\alpha(\pi_G(G))$ is coessential in K because α is an **Arch** epimorphism, it follows that $\theta(k) = 0$ for all $k \in K$. Setting $k = h_1$ gives $\theta(h_1) = 0$, with the result that $\theta(h) = 0$, meaning that G is coessential in H.

It remains to show that for any $v \in G^+$ the induced map $\delta_v: G/v^{\perp} \to H/v^{\perp}$ is a **W** epimorphism. To that end fix v and $h \in H^+$, and for brevity write $\hat{f}$ for the Yosida representation of an element $v^{\perp} + f \in H/v^{\perp}$. Since α is an epimorphism in **Arch** it follows that the induced map from $\pi_G(G)/\pi_G(v)^{\perp}$ to $K/\alpha\pi_G(v)^{\perp} = K/v_1^{\perp}$ is a **W** epimorphism. Thus there is a countable subset $K_0 \subseteq K^+$ such that $\{\check{k}: k \in K_0\}$ is a set of epi indicators for $\check{h}_1$ in $K/v_1^{\perp}$, where $\check{k}$ denotes the Yosida representation of the element $v_1^{\perp} + k \in K/v_1^{\perp}$. Now $K \in \mathbf{B}$ implies $K/v_1^{\perp} \in \mathbf{B}$ by Corollary 3.10, so there is some $k_0 \in K^+$ such that the product of $\check{k}_0$ with $\widetilde{u \wedge v}$ is the constantly one function on $Y(K/v_1^{\perp})$. We claim that $\{\hat{k}: k \in K_0 \cup \{k_0\}\}$ is a set of epi indicators for $\hat{h}$.

To verify the claim consider $(p, q) \in \Sigma(\hat{h})$, which means that $\hat{g}(p) = \hat{g}(q)$ for all $g \in G$, but $\hat{h}(p) \neq \hat{h}(q)$. Now $v_1 \wedge v_2 = 0$ implies that $X_1 = \hat{v}_1^{\leftarrow}(1)$ and $X_2 = \hat{v}_2^{\leftarrow}(1)$ partition $X = Y(H/v^{\perp})$ into disjoint clopen subsets. Furthermore, X_1 can be taken to be the Yosida space of $H/v_1^{\perp}$ and the association $v_1^{\perp} + f \mapsto \hat{f}|_{X_1}$ to be the corresponding representation of its elements. Because the map $v_1^{\perp} + f \mapsto v_1^{\perp} + f_1$ is a **W** isomorphism from $H/v_1^{\perp}$ onto $K/v_1^{\perp}$, there is a homeomorphism $\nu: X_1 \to Y(K/v_1^{\perp})$ (which realizes the inverse of the **W** isomorphism) such that $\hat{f}(t) = \check{f}\nu(t)$ for all $t \in X_1$.

The argument breaks into three cases. If $p, q \in X_1$ then, letting g be an element of G such that $h_2 = g_2$, we would get the contradiction

$$\hat{h}(p) = \hat{h}_2(p) = \hat{g}_2(p) = \hat{g}(p) = \hat{g}(q) = \hat{g}_2(q) = \hat{h}_2(q) = \hat{h}(q).$$

If $p \in X_1$ but $q \in X_2$ then $u \wedge v_2 = 0$ implies $\hat{u}(p) = \hat{u}(q) = 0$. Therefore $(u \widehat{\wedge} v)(p) = 0$, hence $(u \widetilde{\wedge} v)(\nu(p)) = 0$. Since the product of $u \widetilde{\wedge} v$ with $\tilde{k}_0$ is 1, it follows that $\tilde{k}_0 \nu(p) = \infty$ and hence that $\hat{k}_0(p) = \infty$. In the final case we have $p, q \in X_1$. In this case $(\nu(p), \nu(q)) \in \Sigma(\tilde{h}_1)$, so there is some $k \in K_0$ such that $\tilde{k}\nu(p) = \infty$ or $\tilde{k}\nu(q) = \infty$. Therefore $\hat{k}(p) = \infty$ or $\hat{k}(q) = \infty$. This completes the proof. $\square$

We shall employ the symbol $S_E\mathbf{B}$ to designate the full subcategory of **Arch** whose objects are the **Arch** extremal subobjects of **B** objects.

Corollary 4.2. *If* $G \in S_E\mathbf{B}$ *then* $(G/u^\perp, u^\perp + u) \in \mathbf{B}$ *for all* $0 < u \in G$.

Proof. Results 3.3, 3.10, and 4.1. $\square$

An *archimedean kernel* of G is a set of the form $\pi^{\leftarrow}(0)$ for some $\pi: G \to H \in \mathbf{Arch}$, and the archimedean kernel generated by a subset $K \subset G$ is the intersection of the archimedean kernels of G containing K. For example, any polar K (meaning $K = K^{\perp\perp}$) is an archimedean kernel. The subject of archimedean kernels is unavoidable for the present analysis precisely because an **Arch** embedding $\iota: G \to H$ is coessential when and only when the archimedean kernel generated by $\iota(G)$ in H is all of H. Moreover, [BH IV] is the study of those extensions $G \leq H$ in which distinct archimedean kernels of H trace distinctly on G. A fact that is crucial to the present investigation is that the archimedean kernels of divisible abelian ℓ-groups are the convex ℓ-subgroups closed under relative uniform convergence. A sequence $\{g_n : n \in N\} \subset G$ *converges uniformly to* $g \in G$ *with regulator* v, written $g_n \to_v g$, provided that for every positive integer k there is a positive integer m such that $|g - g_n| \leq (1/k)v$ for all $n \geq m$. Now suppose that $G \leq H \in \mathbf{Arch}$; define $G(0)$ to be the convexification of G in H, $G(\alpha + 1)$ to be $\{h \in H : g_n \to_v h$ for some $\{g_n\} \subset G(\alpha)$ and $v \in H\}$, and $G(\beta)$ to be $\bigcup_{\alpha < \beta} G(\alpha)$ for limit ordinals β. The relevant theorem [LZ] states that, for a divisible abelian ℓ-group H with ℓ-subgroup G, the archimedean kernel of H generated by G is $G(\aleph_1)$. Finally, when $0 < u \in G \leq H$ we write $u^{\perp G}$ $(u^{\perp H})$ for $\{x \in G(H) : |x| \wedge u = 0\}$.

In the following proposition we have occasion once again to verify the hypotheses of the Fundamental Theorem for **W**. This time it is convenient to replace the condition $\tau(p) = \tau(q)$ for membership of (p, q) in $\Sigma(h)$ by the condition $\hat{g}(p) = \hat{g}(q)$ foir all $g \in G$. These conditions are clearly equivalent since (the Yosida representation of) G separates the points of $Y(G)$.

Proposition 4.3. *Suppose that* $G \leq H$ *is an* **Arch** *embedding such that every pairwise disjoint countable subset of* G *has a supremum in* H, *and suppose further that* H *is generated as an ℓ-subgroup by those suprema together with the elements of* G. *Then the embedding is* **Arch** *epi.*

Proof. We first show $G \leq H$ coessential. Consider a pairwise disjoint countable set $\{g_n\} \subset G$, let $h_n = \bigvee_1^n g_n \in G$, and let $\bigvee_N h_n = \bigvee_N g_n = h \in H$ and $\bigvee_N (ng_n) = k \in H$. Then $h_n \to_k h$ because, for any $n \geq m$, $m|h - h_n| = m(h - h_n) \leq m(h - h_m) = m \bigvee_{m+1}^\infty g_n = \bigvee_{m+1}^\infty mg_n \leq \bigvee_N (ng_n) = k$.

We next fix $0 < u \in G$ and show $G/u^{\perp G} \leq H/u^{\perp H}$ to be **W** epi as follows. Let $\pi: H \to H/u^{\perp H}$ be the natural map, let Y be the Yosida space of $(H/u^{\perp H}, \pi(u))$, and for each $f \in H$ let $\hat{f}$ be the extended real-valued function which corresponds to $\pi(f)$ by the Yosida representation. Let $\{g_n\}$, h, and k have the meaning of the paragraph above; we claim that $\hat{k}$ epi indicates for $\hat{h}$. To establish this claim consider $x, y \in Y$ such that $\hat{g}(x) = \hat{g}(y)$ for all $g \in G$ and such that $\hat{h}(x) < \hat{h}(y)$. Find positive integers a and b such that $\hat{h}(x) < a/b < \hat{h}(y)$. Define f_n to be $(bg_n - au) \vee 0 \in G$, and let q stand for $\bigvee_N f_n = (bh - au) \vee 0 \in H$. Then $0 = \hat{q}(x) < \hat{q}(y)$, which implies that $0 = \hat{f}_n(x) = \hat{f}_n(y)$ for all n. Let p be $\bigvee_N (nf_n)$. Then for any positive integer m we have $r = (mq - p) \vee 0 = (\bigvee_N (mf_n) - \bigvee_N (nf_n)) \vee 0 = \bigvee_1^m (m - n)f_n$, so $\hat{r}(y) = \bigvee_1^m (m - n)\hat{f}_n(y) = 0$, meaning $m\hat{q}(y) \leq \hat{p}(y)$. Since $p = \bigvee_N (nf_n) = \bigvee_N ((bng_n - anu) \vee 0) \leq \bigvee_N (bng_n) = bk$, we conclude $\hat{k}(y) = \infty$, which proves the claim that $\hat{k}$ epi indicates for $\hat{h}$. The Fundamental Theorem for **Arch** completes the proof of the proposition. $\square$

Corollary 4.4. *Each $S_E\mathbf{B}$ object is laterally σ-complete.*

Proof. Suppose $G \leq B$ is an **Arch** extremal monic embedding of the **Arch** object G into the **B** object B. Now every countable pairwise disjoint subset of G has a supremum in B by virtue of the lateral σ-completeness of B; let H designate the ℓ-subgroup of B generated by G together with all such suprema. Then $G \leq H$ is **Arch** epi by Proposition 4.2, hence $G = H$ because $G \leq B$ is extremal, from which we conclude that G is laterally σ-complete. $\square$

Proposition 4.5. *An archimedean ℓ-group G is conditionally σ-complete if and only if $G/u^{\perp}$ is conditionally σ-complete for each $0 < u \in G$.*

Proof. Suppose that G is conditionally σ-complete, fix $0 < u \in G$ with natural map $\pi: G \to G/u^{\perp}$, and consider $\{g_n\} \subset G$ and $v \in G$ such that $0 \leq \pi(g_n) \leq \pi(v)$ for all n. By replacing each g_n by $(g_n \wedge v) \vee 0$, we may assume that $0 \leq g_n \leq v$. Then $\bigvee_N g_n = g$ exists in G by virtue of its conditional σ-completeness, and $\bigvee_N \pi(g_n) = \pi(g)$ because the kernel of π, namely $u^{\perp}$, is closed.

Now suppose $G/u^{\perp}$ is conditionally σ-complete for each $0 < u \in G$, and consider $\{g_n\} \subset G$ and $u \in G$ such that $0 \leq g_n \leq u$ for all n. Let $\bigvee_N \pi(g_n) = \pi(v)$ in $G/u^{\perp}$, and replace v by $u \wedge v$ to get $v \leq u$. We claim that $\bigvee_N g_n = v$ in G. To establish this claim consider some $f \in G$ such that $g_n \leq f \leq v$ for all n. Then $\pi(g_n) \leq \pi(f) \leq \pi(v)$ for all n implies $v - f \in u^{\perp}$, while $0 \leq v - f \leq v \leq u$ implies $v - f \in u^{\perp\perp}$. Since $u^{\perp} \cap u^{\perp\perp} = 0$, we have $v = f$, which proves the claim and the proposition. $\square$

Proposition 4.6. *If G is laterally σ-complete and has the principal projection property then $G/u^{\perp}$ is also laterally σ-complete for each $0 < u \in G$.*

Proof. $G/u^\perp$ is isomorphic to $u^{\perp\perp}$, so we show $u^{\perp\perp}$ to be laterally σ-complete. A pairwise disjoint countable subset $\{d_n\} \subset u^{\perp\perp}$ has a supremum in G, and this supremum lies in $u^{\perp\perp}$ because of the closure of polars (see the remark preceding Corollary 3.10). $\square$

A *component* of an element $g \in G$ is another element $x \in G$ such that $|x| \wedge |g - x| = 0$. Put another way, x is a component of $g > 0$ if and only if x is an element of some pairwise disjoint set whose supremum is g. For example, the projections $\pi(g)$ and $\psi(g)$ of g on the polars $u^\perp$ and $u^{\perp\perp}$, when they exist, are components of g. If x and y are both components of g then so are $x \vee y$, $x \wedge y$, $x \vee y - y$, and $y - x \wedge y$.

Lemma 4.7. *If a positive element g of an ℓ-group G is a supremum of a countable set of its components, then it is also a supremum of a countable pairwise disjoint set of its components.*

Proof. Suppose $g = \bigvee_N g_n$ for components g_n of g with $g_0 = 0$. Let $k_m = \bigvee_0^m g_n - \bigvee_0^{m-1} g_n$ for $m \geq 1$. The k_m's are clearly components of g; moreover, since $\bigvee_0^{m-1} g_n$ is a component of $\bigvee_0^m g_n$, k_m is disjoint from $\bigvee_0^{m-1} g_n$ and so also from k_p for $p < m$. Since $\{k_n : 1 \leq n \leq m\}$ is pairwise disjoint, its supremum coincides with its sum, which telescopes to $\bigvee_0^m g_n$. We conclude that $\bigvee_N k_n = g$. $\square$

We are adrift in **Arch** without the anchor of the Yosida representation. Thus we resort in the next lemma to the hoary device of showing that two positive elements a and b are disjoint in G by showing that no prime P of G omits $a \wedge b$. We remind the reader that $G(1)$ denotes the first stage in the iterative definition of the relative uniform closure of G (see the discussion preceding Proposition 4.3).

Lemma 4.8. *Suppose H is a divisble archimedean ℓ-group with ℓ-subgroup G having the principal projection property. If for every $0 < u \in G$ the induced map $\delta: G/u^{\perp G} \to H/u^{\perp H}$ is onto, then G is convex in H, and every positive element of $G(1)$ is the supremum of a countable pairwise disjoint subset of G.*

Proof. G is convex in H under these circumstances, for if $h \in H$ satisfies $0 \leq h \leq g \in G$ then there is some $f \in g$ such that $\delta(g^{\perp G} + f) = g^{\perp H} + f = g^{\perp H} + h$. Then the principal projection property of G provides $k \in g^{\perp\perp G}$ such that $f - k \in g^{\perp G}$. We claim that $h = k \in G$. To establish the claim, first observe that $(k - g) \vee 0 \in g^{\perp G}$, for otherwise a prime P of H omitting $((k-g)\vee 0)\wedge g$ must satisfy $P < P+g < P+k$. Then $|k| \wedge |f-k| = 0$ and $k \notin P$ prime imply $f - k \in P$ or $P + k = P + f$; similarly $g \wedge |f - h| = 0$ and $g \notin P$ imply $P+f = P+h$. Combining gives $P+g < P+h$, contrary to the assumption that $h \leq g$, and this shows $(k - g) \vee 0 \in g^{\perp G}$. But since $(k - g) \vee 0 \in g^{\perp\perp G}$ and $g^{\perp G} \cap g^{\perp\perp G} = 0$, we have $k \leq g$. Therefore $k - h \in g^{\perp\perp H}$, and since we already know that $k - h = (k - f) + (f - h) \in g^{\perp H}$ and since $g^{\perp H} \cap g^{\perp\perp H} = 0$, we prove the claim that $k = h$.

Now suppose that $0 \leq g_n \to_v h \in G(1)$ for $\{g_n\} \subset G$ and $v \in H$. For each n let $f_n = \bigvee\{ng_i : 1 \leq i \leq n\} \in G$ and let $d_n = (f_n - h) \vee 0$. Note that $0 \leq d_n \leq f_n$, so that $d_n \in G$ by convexity. Now use the principal projection property to find the projection k_n of $f_n \wedge h$ on $d_n^{\perp\perp G}$, meaning that $k_n \in d_n^{\perp\perp G}$ and $f_n \wedge h - k_n \in d_n^{\perp G}$. We claim that k_n is the projection of h on $d_n^{\perp\perp H}$. To verify this claim we first must check that $k_n \in d_n^{\perp\perp H}$, which is readily done by noting that $d_n^{\perp\perp G} = d_n^{\perp\perp H} \cap G$ by the convexity of G in H. Then we must check that $|h - k_n| \in d_n^{\perp H}$, which is most easily done by contradiction. Suppose P is a prime of H omitting $d_n \wedge |h - k_n|$. Then $d_n \notin P$ implies on the one hand that $P + f_n > P + h$ or $P + f_n \wedge h = P + h$ and on the other that $d_n^{\perp H} \subset P$, and in particular that $f_n \wedge h - k_n \in P$, and this gives $P + f_n \wedge h = P + k_n$. Combining yields $P + h = P + k_n$, contrary to the assumption that $|h - k_n| \notin P$. This proves the claim. The claim, in turn, shows k_n to be a component of h.

To finish the proof it is sufficient by Lemma 4.7 to demonstrate that $h = \bigvee_N k_n$ in H. Suppose for contradiction that there is some $s \in H$ such that $h > s \geq k_n$ for all $n \in N$. Observe that since $\bigvee_N g_n^{\perp\perp H}$ is an archimedean kernel which contains all the g_n's, and since any archimedean kernel is closed with respect to relative uniform convergence, $h^{\perp\perp H} \subset \bigvee_N g_n^{\perp\perp H}$. Note also that $g_n^{\perp\perp H} = \bigvee\{((mg_n - h) \vee 0)^{\perp\perp H} : m \in N\}$, from which it follows that $\bigvee_N d_n^{\perp\perp H} \supset h^{\perp\perp H}$. Therefore there is an integer n such that $0 \neq d_n^{\perp\perp H} \cap (h - s)^{\perp\perp H}$; let P be a prime omitting $d_n \wedge (h - s)$. On the other hand $d_n \notin P$ implies that $d_n^{\perp H} \subset P$ and in particular that $|h - k_n| \in P$, meaning that $P + h = P + k_n$, and on the other hand $h - s \notin P$ implies $P + h > P + s$. Combining gives $P + k_n > P + s$, contrary to the assumption that $s \geq k_n$. This finishes the proof. □

Theorem 4.9. *The following are equivalent for $G \in$ Arch.*
   (a) *G is Arch epicomplete.*
   (b) *$G \in S_E \mathbf{B}$.*
   (c) *$(G/u^\perp, u^\perp + u) \in B$ for all $0 < u \in G$, and G is laterally σ-complete.*
   (d) *G is laterally and conditionally σ-complete, and G is divisible.*

Proof. To show that (a) implies (b), first observe that any $G \in$ **Arch** can be embedded in a **B** object B (as in [B], for example). If G has no proper **Arch** epi extensions of any kind, then it has none in B; therefore any such embedding is an **Arch** extremal monomorphism. (b) implies (c) by Corollaries 4.2 and 4.4, and (c) is equivalent to (d) by results 3.9, 4.5, and 4.6. It remains to show that (c) implies (a).

Suppose that G satisfies (c), let $\iota : G \to H$ be an **Arch** epi embedding, and, to prove ι onto, consider $0 < h \in H$. Since $\iota(G)$ is coessential in H, we may assume that $h \in \iota(G)(1)$. For any $0 < u \in G$ we know that the map $\delta : G/u^\perp \to H/\iota(u)^\perp$ is injective and **W** epi; in fact, it is onto because $G/u^\perp$ is a **B** object and therefore **W** epicomplete by Theorem 3.9. Lemma 4.8 now

shows that h is the supremum in H of some pairwise disjoint countable subset of $\iota(G)$, and this subset must also have a supremum in $\iota(G)$ because of the lateral σ-completeness of G. But suprema in $\iota(G)$ and H agree because of the convexity of the former in the latter. That is, ι is onto and G is **Arch** epicomplete. $\square$

As we have remarked elsewhere (Proposition 5.1 of [BH III]), the notions of **W** epicompleteness and of **Arch** epicompleteness coincide for **W** objects.

Corollary 4.10. *If G is* **W** *epicomplete then G is* **Arch** *epicomplete; if G is* **Arch** *epicomplete and contains the weak unit u, then (G, u) is* **W** *epicomplete.*

Proof. The only substantive difference between condition (d) of Theorem 4.9 and condition (c) of Theorem 3.9 is membership in **Arch** versus membership in **W**. $\square$

It is easy to give an example of an **Arch** epicomplete object which is not **W** epicomplete. Let Y be an uncountable discrete space, and let G be $\{g \in C(Y): \mathrm{coz}(g)$ is countable$\}$. Then G is clearly both divisible and conditionally and laterally σ-complete and is therefore **Arch** epicomplete. However, G has no weak units.

We are now in a position to prove that the **Arch** epicomplete objects form a monoreflective subcategory. Thus the following corollary is parallel to Corollary 3.11. As a result of the nice properties of **Arch**, notably the fact that it is co-well powered [BH II], the composition and product of extremal monomorphisms are again extremal monomorphisms (see Propositions 34.2 and 34.4 of [HS]). Therefore $S_E\mathbf{B}$ is a monoreflective subcategory of **Arch** [HS].

Corollary 4.11. *For each $G \in$* **Arch** *there is an* **Arch** *epicomplete object H_G and* **Arch** *epi injection $\iota_G: G \to H_G$ with the universal mapping property: for any* **Arch** *epicomplete object H and* **Arch** *map $\theta: G \to H$ there is a unique* **Arch** *map $\tau: H_G \to H$ such that $\tau\iota_G = \theta$.*

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